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LETTER TO THE EDITOR

The percolation probability for the site problem on the triangular lattice

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Abstract. The percolation probability for the site problem on the triangular lattice is investigated by series methods. It is concluded that $P(p)$ vanishes near the critical point like $(p-p_c)^\beta$ with $\beta \simeq 0{\cdot}14 \pm 0{\cdot}03$.

A general introduction to the theory of random mixtures and percolation processes, together with a survey of the recent literature, is given in the reviews by Shante and Kirkpatrick (1971) and by Essam (1972). We assume a general familiarity with these problems such as may be derived from these articles. There is a close formal analogy between the mean number and size of finite clusters and the percolation probability in a random mixture on the one hand, and the free energy, initial susceptibility and spontaneous magnetization in a ferromagnet on the other. It has been suggested that random mixtures will exhibit scaling law behaviour (Kasteleyn and Fortuin 1969, Essam 1972); critical exponents are therefore of special theoretical interest. Rudd and Frisch (1970) attempted to estimate critical behaviour from existing Monte Carlo data for percolation probabilities; however they were unable to find a reliable procedure for removing both the systematic and random errors from the data, and reached no firm conclusions. It would seem natural to attempt a precise investigation of critical behaviour along the same general lines that have proved successful with analogous functions for the Ising model; for these most of the evidence on critical indices has been obtained from series expansions and a few exact results.

One important set of critical exponents refers to the high density region $p > p_c$ (Essam 1972); before undertaking a comprehensive investigation we have made a pilot study of the percolation probability for the site problem on the triangular lattice. Our objective has been to determine whether an adequate number of coefficients can be derived and whether their behaviour makes extrapolation possible. We have chosen the site problem on the triangular lattice because the critical concentration is known exactly (Sykes and Essam 1964) and the lattice is well suited to the derivation of series of useful length. We have already found that in the low density region, $p < p_c$, series expansions are in general less well behaved than the corresponding Ising series (Sykes *et al* 1973).

We suppose the sites of the triangular lattice are occupied with probability p. We study the *percolation probability* $P(p)$ defined as the probability that an occupied site, chosen at random, will be connected to infinitely many others; below the *critical concentration* p_c this probability is zero. When p is close to unity the percolation probability may be expanded as a development in powers of $q = 1-p$ by the general methods proposed by Domb (1959) and described by Sykes and Essam (1964) and Essam (1972). We have derived the expansion:

$$P(p) = 1-q^6-6q^8-27q^{10}+6q^{11}-111q^{12}+72q^{13}-534q^{14}+638q^{15}-2868q^{16} +5004q^{17}-17\,408q^{18}+36\,162q^{19}-106\,035q^{20}\ldots \tag{1}$$

It is to be supposed that this expansion is convergent up to some $q' > 0$. From the general behaviour of the coefficients it seems that q' cannot be identified with $q_c = 1-p_c$; a regular alternation in sign suggests a singularity on the *negative* axis. This is confirmed by the Dlog Padé approximants. The closest (and dominant) singularity to the origin $q = 0$ is indicated on the negative real axis at $q^* \simeq -0{\cdot}37 \pm 0{\cdot}06$; the closest singularity on the positive real axis is indicated close to the exact value $q_c = \frac{1}{2}$. There are also indicated a number of pairs of singularities in the complex plane, on or close to the circle $|q| = \frac{1}{2}$, whose position is difficult to estimate with any precision. In tables 1 and 2 we give the diagonal and paradiagonal sequences for the dominant and physical singularities.

On the assumption that near p_c the percolation probability vanishes as a power law we write

$$P(p) \sim B(p-p_c)^{\beta} \tag{2}$$

and form estimates for β from the Padé approximants to $(q-q_c)\,\mathrm{d}\log P/\mathrm{d}q$ evaluated at $q_c = \frac{1}{2}$. We give these in table 3. It seems reasonable to conclude that

$$\beta \simeq 0{\cdot}14 \pm 0{\cdot}03. \tag{3}$$

Table 1. Dlog Padé analysis: closest singularity to the origin. Estimates of q^* (and the corresponding residues) from the Padé approximants to d log P/dq.

n	$[n-1/n]$	$[n/n]$	$[n+1/n]$
5	—	−0·4332 (0·0340)	−0·3921 (0·0108)
6	−0·4083 (0·0174)	−0·4153 (0·0211)	−0·4049 (0·0156)†
7	−0·4102 (0·0183)†	−0·4261 (0·0278)†‡	−0·3558 (0·0027)
8	−0·3891 (0·0101)	−0·3794 (0·0073)	−0·3761 (0·0064)
9	−0·3734 (0·0056)	−0·4130 (0·0127)‡	−0·3737 (0·0058)
10	−0·3710 (0·0051)†		

† Defect on positive axis.
‡ Defect on negative axis.

Table 2. Dlog Padé analysis: closest singularity to the origin on the positive real axis. Estimates of q_c (and the corresponding exponent) from the poles and residues of the Padé approximants to d log P/dq.

n	$[n-1/n]$	$[n/n]$	$[n+1/n]$
6	None	0·46463 (0·0346)	None
7	None	None	None
8	0·45387 (0·0357)	0·50058 (0·1337)	0·46818 (0·0496)
9	0·47951 (0·0703)	0·48555 (0·0844)‡	None
10	0·47448 (0·0615)†		

† Defect on positive axis.
‡ Defect on negative axis.

Table 3. Padé estimates of β from $(q-q_c)\,\mathrm{d}\log P/\mathrm{d}q$ using knowledge that $q_c = \frac{1}{2}$.

n	$[n-1/n]$	$[n/n]$	$[n+1/n]$
5	—	0·1304	0·0674
6	0·0923	0·0731	0·0671†
7	0·0902‡	0·1093§	0·1312
8	0·1681	0·1311	0·1312‡
9	0·1396	0·1458	0·1353§
10	0·1405‡‡		

† Defect on positive axis.
‡ Defect on negative axis.
§ Defect in complex plane.

The limits of uncertainty are of necessity rather wide because of the presence of so many non-physical singularities, especially the dominant one at $q^* \sim -0{\cdot}37$. The hypothesis that $\beta = \frac{1}{8}$, the exact value for the spontaneous magnetization of the two-dimensional Ising model, cannot be excluded. To calculate the function numerically we write

$$P(p) = B^*(q)(q_c-q)^{0\cdot 14} \tag{4}$$

and evaluate Padé approximants to the series for $(q_c-q)[P(p)]^{-1/0\cdot 14}$ in the interval $0 \leqslant q \leqslant q_c$. Figure 1 is based on the [9/9] approximant but other high order approximants give consistent results to within graphical accuracy. The function has the general appearance of a spontaneous magnetization curve with a characteristically sharp cut-off at the critical point. The critical amplitude corresponding to the [9/9] approximant is $B^*(\frac{1}{2}) = 1{\cdot}572$.

From the investigation we have briefly reported we draw the general conclusion that the critical index β is accessible by the method of series expansions and their extrapolation by Padé approximants. The series expansions in the high density region are not very well behaved and do not in general converge up to q_c. A comprehensive

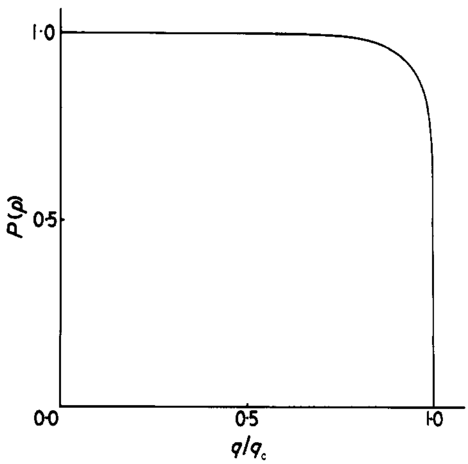

Figure 1. Percolation probability for the site problem on the triangular lattice as a function of q/q_c.

analysis of site and bond problems on a variety of two-dimensional lattices, together with the hypothesis that the index is a dimensional invariant (Shante and Kirkpatrick 1971), seems likely to result in a narrowing of the uncertainties on the value of β. We are undertaking such an analysis; preliminary results for other lattices are consistent with (3) and have not so far conflicted with the hypothesis that β is a dimensional invariant.

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